

# Geometries with Killing Spinors and Supersymmetric $AdS$ Solutions

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## Abstract

The seven and nine dimensional geometries associated with certain classes of supersymmetric  $AdS_3$  and  $AdS_2$  solutions of type IIB and  $D = 11$  supergravity, respectively, have many similarities with Sasaki-Einstein geometry. We further elucidate their properties and also generalise them to higher odd dimensions by introducing a new class of complex geometries in  $2n+2$  dimensions, specified by a Riemannian metric, a scalar field and a closed three-form, which admit a particular kind of Killing spinor. In particular, for  $n \geq 3$ , we show that when the geometry in  $2n+2$  dimensions is a cone we obtain a class of geometries in  $2n+1$  dimensions, specified by a Riemannian metric, a scalar field and a closed two-form, which includes the seven and nine-dimensional geometries mentioned above when  $n = 3, 4$ , respectively. We also consider various ansatz for the geometries and construct infinite classes of explicit examples for all  $n$ .

# 1 Introduction

An interesting class of geometries in seven and nine dimensions was recently discovered in [1, 2] and further explored in [3]. The geometries are specified by a Riemannian metric, a scalar field  $B$  and a closed two-form  $F$  and they admit Killing spinors of a certain type. The seven dimensional geometries give rise to supersymmetric solutions of type IIB supergravity with a three dimensional anti-de-Sitter space ( $AdS_3$ ) factor and these are dual to supersymmetric conformal field theories (SCFTs) with  $(0, 2)$  supersymmetry in two-dimensions. Similarly, the nine-dimensional geometries give rise to supersymmetric solutions of  $D = 11$  supergravity with  $AdS_2$  factors and these are dual to superconformal quantum mechanics with two supercharges.

This geometry in  $2n + 1$  dimensions, with  $n = 3, 4$ , is strikingly similar to Sasaki-Einstein geometry. In particular, they both have a Killing Reeb vector of constant norm and define a  $U(n)$  or metric contact structure. The Killing vector defines a natural foliation and in the Sasaki-Einstein case the metric transverse to these orbits is Kähler and Einstein, while for the geometry considered in [1, 2] it is Kähler and in addition satisfies

$$\square R - \frac{1}{2}R^2 + R_{ij}R^{ij} = 0, \quad (1.1)$$

where  $R$  and  $R_{ij}$  are the Ricci-scalar and Ricci-tensor, respectively, of the transverse metric and we also demand<sup>1</sup> that  $R > 0$ . Moreover, locally, the whole geometry can be reconstructed from a local Kähler metric in  $2n$ -dimensions satisfying (1.1).

Recall that a succinct definition of a Sasaki-Einstein metric in  $2n + 1$  dimensions is that the corresponding cone metric in  $2n + 2$  dimensions, with base given by the Sasaki-Einstein metric, is Ricci-flat and Kähler, or equivalently has  $SU(n + 1)$  holonomy. A metric with  $SU(n + 1)$  holonomy has an  $SU(n + 1)$  structure, specified by a fundamental two-form  $J$  and an  $(n + 1, 0)$ -form  $\Omega$  (which together define a metric), with vanishing intrinsic torsion,

$$dJ = d\Omega = 0. \quad (1.2)$$

Equivalently it can be characterised as admitting covariantly constant spinors.

In this paper we will determine the analogous statements for the geometries studied in [1, 2] and furthermore generalise the geometry from  $n = 3, 4$  to all  $n \geq 3$ . We find that the analogue of a metric with  $SU(n + 1)$  holonomy is a geometry specified

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<sup>1</sup>Note that when  $R < 0$  we can construct Lorentzian geometries which for  $n = 3, 4$  give rise to solutions of type IIB supergravity and  $D = 11$  supergravity with an  $S^3$  and  $S^2$  factor, respectively, as described in [3].

by a metric, a scalar field  $\phi$  and a closed three-form  $f$  with an  $SU(n+1)$  structure  $(J, \Omega)$  satisfying

$$\begin{aligned} d[e^{n\phi}\Omega] &= 0 \\ d[e^{2(n-1)\phi}J^n] &= 0 \\ d[e^{2\phi}J] &= f \end{aligned} \tag{1.3}$$

and in addition

$$d[e^{2(n-3)\phi} *_{2n+2} f] = 0, \tag{1.4}$$

where  $*_{2n+2}$  is the Hodge dual using the  $(2n+2)$ -dimensional metric defined by the  $SU(n+1)$  structure. Note that (1.3) imply that the almost complex structure associated with the  $SU(n+1)$  structure is integrable. We will show that complex geometries satisfying (1.3) are equivalent to geometries that admit a certain type of Killing spinor. Furthermore, by analysing the integrability conditions for the Killing spinor equations, and in addition imposing (1.4), we will also determine the equations of motion satisfied by the metric, the scalar and the three-form, which are the analogue of the property of Ricci-flatness in the case of  $SU(n+1)$  holonomy.

If we demand that the geometry in  $2n+2$  dimensions satisfying (1.3), (1.4) is a metric cone and with the scalar and the three-form having a specific scaling:

$$\begin{aligned} ds_{2n+2}^2 &= dr^2 + r^2 ds_{2n+1}^2 \\ e^{-2\phi} &= r^{\frac{2(n-1)}{n-2}} e^B \\ f &= r^{\frac{n}{2-n}} dr \wedge F, \end{aligned} \tag{1.5}$$

we will show that we obtain a geometry in  $2n+1$  dimensions specified by a metric, with line element  $ds_{2n+1}^2$ , a scalar field,  $B$ , and a closed two-form,  $F$ , all independent of the co-ordinate  $r$ , which for  $n=3,4$  is precisely equivalent to the geometry of [1, 2]. We show that for all  $n$  the  $2n+1$  dimensional metric has a Killing vector of constant norm and that the transverse metric is Kähler and satisfies (1.1). For all  $n$ , locally, the whole geometry can be reconstructed from a local Kähler metric in  $2n$ -dimensions satisfying (1.1). We determine the kind of Killing spinors that the geometry in  $2n+1$  dimensions admit and also the kind of equations of motion that are satisfied by the metric, the scalar field  $B$  and the two-form  $F$ , which are the analogue of the Einstein condition in the case of Sasaki-Einstein geometry.

Note that in  $2n+2$  dimensions we are generalising the notion of  $SU(n+1)$  holonomy in the sense that if we set  $f = \phi = 0$  in (1.3), (1.4) we clearly return to the case of  $SU(n+1)$  holonomy. However, on the base of the cone in  $2n+1$  dimensions,

defined by (1.5), we are not generalising the notion of Sasaki-Einstein geometry: for example (1.1) is not satisfied for Einstein metrics for  $n \geq 3$ .

When  $n = 3$  an eight dimensional geometry satisfying (1.3), (1.4) gives rise to a supersymmetric solution of type IIB supergravity which is a warped product of  $\mathbb{R}^{1,1}$  with the eight-dimensional geometry. Assuming that the eight dimensional geometry is a cone as in (1.5) we recover the type IIB  $AdS_3$  solutions of [1]. Similarly, as discussed in [4], when  $n = 4$ , a ten-dimensional geometry satisfying (1.3), (1.4) gives rise to a supersymmetric solution of  $D = 11$  supergravity which is a warped product of  $\mathbb{R}$  with the ten-dimensional geometry. If the ten dimensional geometry is a cone as in (1.5) we recover the  $D = 11$   $AdS_2$  solutions of [2].

We do not know of any physical application for the geometry when  $n \geq 5$ . However, it is possible that the geometries in  $2n + 1$  dimensions, with  $n \geq 5$ , inherit some properties dictated by physics for the seven and/or nine dimensional cases. This is by analogy with the Sasaki-Einstein case. Recall that five-dimensional Sasaki-Einstein geometries,  $SE_5$ , give rise to  $AdS_5 \times SE_5$  solutions of type IIB supergravity. These solutions are dual to  $N = 1$  supersymmetric conformal field theories (SCFTs) in four spacetime dimensions and such SCFTs exhibit the phenomenon of  $a$ -maximisation [5]. Motivated by this observation, it was proven in [6, 7] that the volume of Sasaki-Einstein manifolds in *any dimension* satisfies a variational principle.

Sections 2 and 3 of this paper will be devoted to expanding on the above discussion. In the subsequent sections we will then consider various ansatz in order to find explicit examples. In section 4 we will construct explicit examples of the geometries in  $2n + 1$  dimensions. This is a direct analogue of the explicit construction of Sasaki-Einstein metrics that was carried out in [8, 9] and generalises the analysis of [3] from  $n = 3, 4$  to all  $n \geq 3$ . More specifically, we construct explicit local Kähler metrics in  $2n$ -dimensions satisfying (1.1), by considering local metrics on line bundles over positively curved Kähler-Einstein manifolds in  $2n - 2$  dimensions. We then argue that for each choice of Kähler-Einstein manifold these lead to countably infinite classes of smooth, compact and simply connected globally defined geometries in  $2n + 1$  dimensions.

Section 5 will present an ansatz for geometries in  $2n + 2$  dimensions that depend on a number of functions of one variable. We show that the ansatz includes the simple case of a cone over a  $2n + 1$  dimensional geometry with the corresponding  $2n$ -dimensional Kähler manifold satisfying (1.1) being a product of Kähler-Einstein spaces. Such  $2n + 1$ -dimensional geometries, for  $n = 3, 4$  were studied in [3]. We also show that the ansatz includes singular non-compact Calabi-Yau geometries, some of which were discussed in [10]. For  $n = 3$  the ansatz also incorporates a known solution

in type IIB supergravity that describes an interpolation between a solution with an  $AdS_5$  factor and a solution with an  $AdS_3 \times H_2/\Gamma$  factor, where  $H_2$  is the hyperbolic plane and  $\Gamma$  is a discrete group of isometries [11, 12]. Similarly, for  $n = 4$  the ansatz covers a known solution in  $D = 11$  supergravity that describes an interpolation between a solution with an  $AdS_4$  factor and a solution with an  $AdS_2 \times H_2/\Gamma$  factor [13, 14].

In section 6 we will consider an ansatz for the geometries in  $2n + 1$  dimensions which is inspired by the work of [15]. The resulting system boils down to solving a differential equation for a function  $D$  of three variables,  $x^1, x^2, z$ . For  $n = 3$  the equation is linear, as in [15], and can be explicitly solved. For  $n \geq 4$  the equation is

$$\Delta D + z^{\frac{n-4}{n-3}} \partial_z^2 e^D = 0, \quad (1.6)$$

where  $\Delta = \partial_1^2 + \partial_2^2$ . For  $n = 4$  this is equivalent to the continuous Toda equation as in [15]. We don't know whether the equation is an integrable system for  $n \geq 5$ .

Section 7 briefly concludes.

## 2 Geometry in $2n + 2$ Dimensions

The geometry in  $2n + 2$  dimensions that we will be interested in is specified by a Riemannian metric,  $g$ , a scalar field,  $\phi$ , and a closed three-form,  $f$ :

$$df = 0. \quad (2.1)$$

We are interested in such geometries that admit a solution to the Killing spinor equations:

$$\begin{aligned} \left[ \gamma^\alpha \nabla_\alpha \phi + \frac{i}{12} e^{-2\phi} f_{\sigma_1 \sigma_2 \sigma_3} \gamma^{\sigma_1 \sigma_2 \sigma_3} \right] \epsilon &= 0 \\ \left[ \nabla_\alpha - \frac{i}{24} e^{-2\phi} f_{\sigma_1 \sigma_2 \sigma_3} \gamma_\alpha^{\sigma_1 \sigma_2 \sigma_3} \right] \epsilon &= 0, \end{aligned} \quad (2.2)$$

where  $\epsilon$  is a  $Spin(2n + 2)$  spinor, the gamma-matrices,  $\gamma^\alpha$ , generate the Clifford algebra  $Cliff(2n + 2)$ :  $\{\gamma_\alpha, \gamma_\beta\} = 2g_{\alpha\beta}$  and the indices  $\alpha, \sigma, \dots$  run from 1 to  $2n + 2$ . We will be particularly interested in geometries admitting such Killing spinors that in addition satisfy the following equation of motion for  $f$ :

$$d \left[ e^{2(n-3)\phi} *_{2n+2} f \right] = 0. \quad (2.3)$$

We now argue that any geometry satisfying (2.1), (2.2), (2.3) is also a solution to the following equations:

$$\begin{aligned} E_{\alpha\beta} &= 0 \\ \nabla^2\phi + 2(n-1)(\nabla\phi)^2 - \frac{1}{2}e^{-4\phi}f^2 &= 0, \end{aligned} \quad (2.4)$$

where we have defined

$$E_{\alpha\beta} \equiv R_{\alpha\beta} - 2(n-1)\nabla_{\alpha\beta}\phi + 2(n-2)\nabla_{\alpha}\phi\nabla_{\beta}\phi + \frac{1}{4}e^{-4\phi}f_{\alpha\sigma_1\sigma_2}f_{\beta}^{\sigma_1\sigma_2} - \frac{1}{2}g_{\alpha\beta}e^{-4\phi}f^2. \quad (2.5)$$

To see this we follow an argument of [16]. Specifically, the integrability conditions for the Killing spinor equations can be used to show that

$$E_{\beta\sigma}\gamma^{\sigma}\epsilon = -\frac{i}{48}e^{-2\phi}df_{\sigma_1\sigma_2\sigma_3\sigma_4}\gamma_{\beta}^{\sigma_1\sigma_2\sigma_3\sigma_4}\epsilon - \frac{i}{4}e^{2(2-n)\phi}\nabla_{\alpha}(e^{-2(3-n)\phi}f^{\alpha}_{\sigma_1\sigma_2})\gamma_{\beta}^{\sigma_1\sigma_2}\epsilon \quad (2.6)$$

and

$$\begin{aligned} &\left[\nabla^2\phi + 2(n-1)(\nabla\phi)^2 - \frac{1}{2}e^{-4\phi}f^2\right]\epsilon = \\ &-\frac{i}{48}e^{-2\phi}df_{\sigma_1\sigma_2\sigma_3\sigma_4}\gamma^{\sigma_1\sigma_2\sigma_3\sigma_4}\epsilon - \frac{1}{4}ie^{2(2-n)\phi}\nabla_{\alpha}(e^{-2(3-n)\phi}f^{\alpha}_{\sigma_1\sigma_2})\gamma^{\sigma_1\sigma_2}\epsilon. \end{aligned} \quad (2.7)$$

If we now impose (2.1), (2.3), we immediately deduce from (2.7) that the scalar equation of motion in (2.4) is satisfied. From (2.6) we similarly deduce that  $E_{\alpha\beta}\gamma^{\beta}\epsilon = 0$ , but on a Riemannian manifold this implies that  $E_{\alpha\beta} = 0$ .

Observe that (2.3), (2.4) are equations of motion that can be derived by varying an action with Lagrangian density given by

$$\mathcal{L}_{2n+2} = e^{2(n-1)\phi} \left[ R + 2n(2n-3)(\nabla\phi)^2 + \frac{1}{2}e^{-4\phi}f^2 \right]. \quad (2.8)$$

Here we have defined  $f^2 \equiv (1/3!)f_{\alpha_1\alpha_2\alpha_3}f^{\alpha_1\alpha_2\alpha_3}$  and we are thinking of the action as being a functional of the metric, the scalar  $\phi$  and a two-form potential  $b$  with  $f = db$ .

We next observe that the only compact solutions to the equations of motion (2.3), (2.4) are Ricci-flat manifolds. To see this note that the scalar equation of motion implies that

$$\nabla^2[e^{2(n-1)\phi}] = 2(n-1)e^{2(n-3)\phi}f^2. \quad (2.9)$$

Integrating this over a compact manifold we deduce for  $n \geq 2$  that  $f = 0$ . The scalar equation of motion in (2.4) then implies that  $\phi = 0$ . A similar argument works for  $n = 1$  also. In section 3 we will focus on non-compact cone geometries which can have compact base spaces.

## 2.1 $SU(n+1)$ structure

We now restrict our considerations to solutions of the Killing spinor equations (2.2) where the Killing spinor  $\epsilon$  is a Weyl spinor. More specifically, we demand that  $\epsilon$  is no-where vanishing and has isotropy group  $SU(n+1) \subset Spin(2n+2)$ . In other words we demand that the Killing spinor fixes a globally defined  $SU(n+1)$ -structure.

We first observe that the Killing spinor equations (2.2) imply that  $\bar{\epsilon}\epsilon$  is a constant. We will fix the normalisation by imposing  $\bar{\epsilon}\epsilon = 1$ . The  $SU(n+1)$  structure is specified by a fundamental two-form  $J$  and an  $(n+1, 0)$ -form  $\Omega$  both of which can be constructed as bi-linears in  $\epsilon$ :

$$\begin{aligned} J_{\alpha\beta} &= -i\bar{\epsilon}\gamma_{\alpha\beta}\epsilon \\ \Omega_{\alpha_1\dots\alpha_{n+1}} &= \bar{\epsilon}^c\gamma_{\alpha_1\dots\alpha_{n+1}}\epsilon, \end{aligned} \tag{2.10}$$

where  $\epsilon^c$  is the spinor conjugate to  $\epsilon$ . Recall that  $(J, \Omega)$  define a metric and an almost complex structure. After some detailed calculations, we find that the Killing spinor equations (2.2) imply that the  $SU(n+1)$  structure must satisfy

$$\begin{aligned} d[e^{n\phi}\Omega] &= 0 \\ d[e^{2(n-1)\phi}J^n] &= 0 \\ d[e^{2\phi}J] &= f. \end{aligned} \tag{2.11}$$

These equations account for all of the intrinsic torsion modules of the  $SU(n+1)$  structure. In particular, using the notation of [17], the first equation in (2.11) says that the torsion modules  $W_1 = W_2 = 0$ , which implies that the manifold is complex (i.e. that the almost complex structure is integrable), and that the Lee form  $W_5 \propto d\phi$ . The second equation in (2.11) says that the Lee form  $W_4 \propto d\phi$ . The third equation in (2.11) relates  $W_3$  and  $W_4$  to the three-form  $f$ .

We have argued that (2.11) are necessary conditions for solutions of the Killing spinor equations (2.2) with spinors that define an  $SU(n+1)$  structure. They are also sufficient. In particular given an  $SU(n+1)$  structure satisfying the first two conditions in (2.11), one can extract  $d\phi$  from the torsion modules  $W_4$  or  $W_5$  and obtain a three-form  $f$  via the last equation. Following the same type of argument as that discussed after equation (4.23) of [16] we conclude that there will be an  $SU(n+1)$  invariant Weyl spinor that solves the Killing spinor equations (2.2).

Clearly the Bianchi identity for  $f$ , (2.1), is automatically implied by (2.11). Thus in light of the integrability argument made in the previous subsection, if we also impose the equation of motion for  $f$ , (2.3), then we deduce that all of the equations

of motion (2.4) are satisfied. Also observe that we are describing a generalisation of manifolds with special holonomy  $SU(n+1)$ . In particular, if  $\phi = f = 0$ , we are demanding the existence of  $SU(n+1)$  invariant covariantly constant spinors, in other words geometries with  $SU(n+1)$  holonomy, and (2.11) reduces to the usual conditions  $dJ = d\Omega = 0$ .

The geometries with Killing spinors that we are describing generalise a certain class of supersymmetric solutions of type IIB and  $D = 11$  supergravity. Specifically, we have checked<sup>2</sup> that the geometry with  $n = 3$  satisfying (2.11) and (2.3) gives rise to a supersymmetric solution of type IIB supergravity of the form:

$$\begin{aligned} ds^2 &= e^\phi [ds^2(\mathbb{R}^{1,1}) + ds_8^2] \\ F_5 &= -\frac{1}{4}[Vol(\mathbb{R}^{1,1}) \wedge f - *_8 f], \end{aligned} \quad (2.12)$$

where  $F_5$  is the self-dual five form. These solutions preserve  $(0, 2)$  supersymmetry with respect to  $\mathbb{R}^{1,1}$ . Similarly, the  $n = 4$  geometry satisfying (2.11) and (2.3) gives [4] the following supersymmetric solution of  $D = 11$  supergravity

$$\begin{aligned} ds^2 &= e^{4\phi/3}[-dt^2 + ds_{10}^2] \\ G_4 &= dt \wedge f. \end{aligned} \quad (2.13)$$

These solutions preserve two supercharges. For both of these cases flux quantisation in the supergravity theory implies that the periods of  $f$  should be rational<sup>3</sup>. One might consider demanding that this condition holds for general  $n$ .

In the next section we will assume that the metric in  $2n+2$  dimensions is a metric cone, as well as imposing additional assumptions on  $\phi, f$ , and study the corresponding geometry on the  $2n+1$ -dimensional base of the cone. Before doing that, let us conclude with two comments which will play no role in the sequel. Firstly, we observe that the Killing spinor equations in (2.2), for arbitrary spinor, can be equivalently written:

$$\begin{aligned} \gamma^\alpha \nabla_\alpha \phi \epsilon + \frac{i}{12} e^{-2\phi} f_{\sigma_1 \sigma_2 \sigma_3} \gamma^{\sigma_1 \sigma_2 \sigma_3} \epsilon &= 0 \\ \left[ \nabla_\alpha + \frac{1}{2} \nabla_\alpha \phi + \frac{1}{2} \nabla_\beta \phi \gamma_\alpha^\beta + \frac{i}{8} e^{-2\phi} f_{\alpha \sigma_1 \sigma_2} \gamma^{\sigma_1 \sigma_2} \right] \epsilon &= 0. \end{aligned} \quad (2.14)$$

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<sup>2</sup>It was recently shown in [18] that this result can also be obtained by considering a restricted class of solutions analysed in [19].

<sup>3</sup>Actually, to be more precise, the quantisation condition on the four-form is slightly different: see [20].



If we now introduce the rescaled metric  $\tilde{g} = e^{2\phi}g$  and the rescaled spinor  $\tilde{\epsilon} = e^{\phi/2}\epsilon$  the Killing spinor equations become (dropping tildes)

$$\begin{aligned}\gamma^\alpha \nabla_\alpha \phi \epsilon + \frac{i}{12} f_{\sigma_1 \sigma_2 \sigma_3} \gamma^{\sigma_1 \sigma_2 \sigma_3} \epsilon &= 0 \\ \nabla_\alpha \epsilon + \frac{i}{8} f_{\alpha \sigma_1 \sigma_2} \gamma^{\sigma_1 \sigma_2} \epsilon &= 0.\end{aligned}\tag{2.15}$$

Interestingly these are just the Killing spinor equations that arise in the common NS-NS sector of supergravity (see [21, 22] and e.g. [23, 17]) with imaginary 3-form flux  $H = if$  and dilaton  $\Phi = \phi$ .

Although we will be focussing on  $n \geq 3$  in the remainder, the second comment concerns the  $n = 2$  case. If we let  $\epsilon$  be a chiral spinor:  $i\gamma_7 \epsilon = \epsilon$  where  $\gamma_7 = \gamma_{123456}$  and define  $H = e^{-2\phi} *_6 f$  and the dilaton  $\Phi = -\phi$ , then we find the Killing spinor equations are exactly the same as in the common NS-NS sector in six dimensions.

### 3 Geometry in $2n + 1$ Dimensions

We now restrict our considerations to  $n \geq 3$  and take the  $2n + 2$  dimensional metric of the last section to be a cone metric:

$$ds_{2n+2}^2 = dr^2 + r^2 ds_{2n+1}^2, \tag{3.1}$$

where  $ds_{2n+1}^2$  is independent of  $r$ . We also demand that the scalar field  $e^{2\phi}$  and the three-form  $f$  have the following dependence on  $r$

$$\begin{aligned}e^{-2\phi} &= r^{\frac{2(n-1)}{n-2}} e^B \\ f &= r^{\frac{n}{2-n}} dr \wedge F.\end{aligned}\tag{3.2}$$

where the scalar field  $B$  and the closed two-form  $F$  are independent of  $r$ . We are interested in the geometry in  $2n + 1$  dimensions on the link  $L$ , defined to be the surface  $r = 1$  on the cone, with metric whose line element is  $ds_{2n+1}^2$ , scalar field  $B$  and closed two-form  $F$ .

We first observe that the equations of motion in  $2n + 2$  dimensions given in (2.1), (2.4) give rise to the following equations of motion in  $2n + 1$  dimensions

$$\begin{aligned}R_{ab} + (n-1)\nabla_{ab}B + \frac{(n-2)}{2}\nabla_a B \nabla_b B + g_{ab}\frac{2}{n-2} + \frac{1}{2}e^{2B}F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F^2 &= 0 \\ \nabla^2 B - (n-1)(\nabla B)^2 - \frac{4(n-1)}{(n-2)^2} + \frac{e^{2B}}{2}F^2 &= 0 \\ d[e^{(3-n)B} *_ {2n+1} F] &= 0\end{aligned}\tag{3.3}$$

where  $F^2 \equiv F_{ab}F^{ab}$ . These equations of motion can be derived from an action with Lagrangian given by<sup>4</sup>

$$\mathcal{L}_{2n+1} = e^{(1-n)B} \left[ R + \frac{n(2n-3)}{2} (\nabla B)^2 + \frac{1}{4} e^{2B} F^2 - \frac{2n}{(n-2)^2} \right]. \quad (3.4)$$

Here we are thinking of the action as being a functional of the metric, the scalar  $B$  and a one-form potential  $A$  with  $F = dA$ .

The Killing spinor equations in  $2n+2$  dimensions (2.2) give rise to Killing spinor equations in  $2n+1$  dimensions. The generators  $\gamma_\alpha$  of  $Cliff(2n+2)$  can be written

$$\begin{aligned} \gamma_a &= \Gamma_a \otimes \sigma_1 \quad a = 1, \dots, 2n+1 \\ \gamma_r &= 1 \otimes \sigma_2, \end{aligned} \quad (3.5)$$

where  $\Gamma_a$  generate  $Cliff(2n+1)$  and  $\sigma_1, \sigma_2$  are Pauli matrices. For definiteness, when  $n$  is odd we take  $\Gamma_1 \dots \Gamma_{2n+1} = -i$  and the chirality operator in  $2n+2$  dimensions as  $\gamma_1 \dots \gamma_{2n+1} \gamma_r = 1 \otimes \sigma_3$ . When  $n$  is even we take  $\Gamma_1 \dots \Gamma_{2n+1} = -1$  and the chirality operator in  $2n+2$  dimensions as  $i\gamma_1 \dots \gamma_{2n+1} \gamma_r = 1 \otimes \sigma_3$ . In both cases, then, a positive chirality spinor in  $2n+2$  dimensions can be written as  $\epsilon = (\eta, 0)$  where  $\eta$  is a spinor in  $2n+1$  dimensions. We then find that substituting (3.1) and (3.2) into (2.2) leads to

$$\begin{aligned} \left[ \Gamma^a \nabla_a B + i \frac{2(n-1)}{n-2} + \frac{1}{2} e^B F_{ab} \Gamma^{ab} \right] \eta &= 0 \\ \left[ \nabla_c + \frac{i}{2} \Gamma_c + \frac{1}{8} e^B F_{ab} \Gamma_c^{ab} \right] \eta &= 0. \end{aligned} \quad (3.6)$$

Using a result of the last section, we also conclude that if we have a solution to these Killing spinor equations and in addition we impose the Bianchi identity,  $dF = 0$ , and the equation of motion for the two-form

$$d \left[ e^{(3-n)B} *_{2n+1} F \right] = 0, \quad (3.7)$$

then all of the equations of motion in (3.3) will be satisfied.

For the  $n = 3$  case a solution to the Killing spinor equations (3.6) and (3.7) give rise to a supersymmetric type IIB solution with an  $AdS_3$  factor of the form

$$\begin{aligned} ds^2 &= e^{-B/2} [ds^2(AdS_3) + ds_7^2] \\ F_5 &= -\frac{1}{4} [Vol(AdS_3) \wedge F - *_7 F], \end{aligned} \quad (3.8)$$

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<sup>4</sup>Note for  $n = 3$  that if we change the sign of the last two terms in this Lagrangian, we obtain a Lagrangian equivalent to one considered in [25].

while for the  $n = 4$  case we can obtain the following solution of  $D = 11$  supergravity with an  $AdS_2$  factor

$$\begin{aligned} ds^2 &= e^{-2B/3} [ds^2(AdS_2) + ds_9^2] \\ G_4 &= Vol(AdS_2) \wedge F. \end{aligned} \quad (3.9)$$

In making the comparison with [1, 2, 3] one should identify  $B = \frac{2(n-1)}{2-n}A$  and in the IIB case we have  $F^{here} = -4F^{there}$ .

We can analyse the geometries in  $2n + 1$  dimensions that admit solutions to the Killing spinor equations (2.2) with  $SU(n + 1)$  invariant spinors in two ways. We can either directly analyse the Killing spinor equations (3.6), generalising the analysis of [1, 2], or equivalently, we can reduce the  $SU(n + 1)$  structure in  $2n + 2$  dimensions satisfying (2.11) that we discussed in the last section. Let us consider the latter approach.

We first stay on the cone. We introduce the Reeb vector field  $\xi$  defined by

$$\xi^\alpha = J^\alpha{}_\beta (r \partial_r)^\beta, \quad (3.10)$$

which has norm squared given by  $r^2$ . In this expression  $J$  refers to the integrable complex structure on the cone obtained by raising an index on the two-form  $J$ ; we hope that using the same letter for both does not cause confusion. We also define the one form  $\eta$  on the cone:

$$\eta_\alpha = J_\alpha{}^\beta \left( \frac{dr}{r} \right)_\beta, \quad (3.11)$$

i.e.  $\eta = \frac{1}{r^2} g_{2n+1}(\xi, \cdot)$ . We will restrict our considerations to no-where vanishing  $e^B$ . Compatible with the cone metric (3.1), we can decompose the  $SU(n + 1)$  structure  $J, \Omega$  via

$$\begin{aligned} J &= r\eta \wedge dr + r^2 e^B J_T \\ \Omega &= r^n e^{nB/2} (dr - ir\eta) \wedge \bar{\Omega}_T, \end{aligned} \quad (3.12)$$

where  $J_T$  is a two-form and  $\bar{\Omega}_T$  is an  $n$ -form both orthogonal to  $\xi$  and  $\partial_r$ . The conditions on the  $SU(n + 1)$  structure (2.11) now become

$$\begin{aligned} dJ_T &= 0 \\ (J_T)^n \wedge de^B &= 0 \\ -\frac{1}{c} e^B (J_T)^n + n(J_T)^{n-1} d\eta &= 0 \\ d\bar{\Omega}_T &= \frac{i}{c} \eta \wedge \bar{\Omega}_T, \end{aligned} \quad (3.13)$$

with the two-form  $F$  given by

$$F = -\frac{1}{c}J_T + d(e^{-B}\eta) \quad (3.14)$$

and we have introduced the constant  $c = (n-2)/2$ . The Bianchi identity for  $F$  is automatically satisfied and so we just need to impose the equation of motion for  $F$ , (3.7), to ensure that all equations of motion (3.3) are satisfied. From these conditions one can show that

$$\mathcal{L}_\xi e^B = \mathcal{L}_\xi J_T = \mathcal{L}_\xi \eta = 0, \quad \mathcal{L}_\xi \bar{\Omega}_T = \frac{i}{c} \bar{\Omega}_T. \quad (3.15)$$

From this we deduce that

$$\mathcal{L}_\xi J = 0, \quad \mathcal{L}_\xi \Omega = \frac{i}{c} \Omega \quad (3.16)$$

and hence that  $\xi$  is Killing and holomorphic. One can also show that  $r\partial_r$  is holomorphic.

Let us now consider the link  $L$ , defined as  $r = 1$ . The vector field  $\xi$  restricts to a vector field on  $L$ , which we denote by the same letter. Similarly, we find that  $\eta$  and  $J$  pull back to well defined forms on  $L$ . One has to be a little careful about  $\bar{\Omega}_T$ : it does not give rise to an  $n$ -form but rather a section of  $\Lambda^n(T^*X)$  twisted by the complex line bundle defined by  $dr - ir\eta$ . For this reason we have a  $U(n)$  structure on  $L$  (or a metric contact structure - see [24] definition 5).

We now introduce local coordinates on  $L$  so that we can write  $\xi = (1/c)\partial_z$ ,  $\eta = c(dz + P)$  and

$$ds_{2n+1}^2 = c^2(dz + P)^2 + e^B ds_{2n}^2. \quad (3.17)$$

Using (3.13) we deduce that  $ds_{2n}^2$  is a local Kähler metric with Kähler-form  $J_T$ . Furthermore, defining  $\Omega_T = e^{-iz}\bar{\Omega}_T$ , we have that  $\Omega_T$  is the local  $(n,0)$ -form on the Kähler manifold satisfying  $d\Omega_T = iP \wedge \Omega_T$  and  $dP = \rho$ , where  $\rho$  is the Ricci-form of the Kähler metric. We also write the Ricci tensor of this Kähler metric as  $R_{ij}$  and the Ricci scalar as  $R$ . The scalar field and the two-form then take the form

$$\begin{aligned} e^B &= c^2 \left( \frac{R}{2} \right) \\ F &= -\frac{1}{c}J_T + cd[e^{-B}(dz + P)]. \end{aligned} \quad (3.18)$$

The equation of the two-form in (3.7) implies that the Kähler metric must satisfy

$$\square R + R_{ij}R^{ij} - \frac{1}{2}R^2 = 0. \quad (3.19)$$

At this point it is worth pausing to emphasise that we have shown that if we have a local Kähler metric in  $2n$ -dimensions that solves this master equation<sup>5</sup> we can reconstruct a local  $2n + 1$ -dimensional geometry via (3.17), (3.18) which admits solutions to the Killing spinor equations (3.6) and solves the equations of motion (3.3).

Returning to the cone, in terms of this local description the original  $SU(n + 1)$  structure  $J, \Omega$  is given by

$$\begin{aligned} J &= -crdr \wedge (dz + P) + r^2 e^B J_T \\ \Omega &= e^{iz} (e^{B/2} r)^n [dr - irc(dz + P)] \wedge \Omega_T \end{aligned} \quad (3.20)$$

and one can directly check that this  $SU(n + 1)$  structure satisfies (2.11). One can also directly check that the equation of motion (2.3) is also satisfied: to do so observe that

$$e^{2(n-3)\phi} *_{2n+2} d[e^{2\phi} J] = -c^2 \left[ \frac{(J_T)^{n-2}}{(n-2)!} \wedge \rho \wedge (dz + P) + *_{2n} d\left(\frac{R}{2}\right) \right]. \quad (3.21)$$

Note that the natural orientation on the cone is given by

$$\frac{(J)^{n+1}}{(n+1)!} = -cr^{2n+1} e^{nB} dr (dz + P) \frac{(J_T)^n}{n!} \quad (3.22)$$

and hence we take  $\epsilon_{rzi_1 \dots i_{2n}} = -1$  and we also take  $\epsilon_{zi_1 \dots i_{2n}} = +1$ .

Since the Killing vector  $\xi$  is no-where vanishing it defines a foliation. Just as in the case of Sasaki structures there are three cases to consider. The regular case is when the orbits of  $\xi$  close and the circle action is free. In this case, the local description above is globally defined. In particular, it is characterised by Kähler manifolds satisfying (3.19). The quasi-regular case is when the orbits of  $\xi$  close but the action is only locally free. In this case the orbit space is an orbifold and  $L$  is the total space of an orbifold circle bundle over a Kähler orbifold satisfying (3.19). The irregular case is when the orbits generically do not close and there is no globally defined Kähler geometry.

In the remaining sections we will illustrate the geometry we have introduced both in  $2n + 1$  dimensions and  $2n + 2$  dimensions by discussing several ansatz, some of which lead to new explicit examples.

## 4 Fibration construction using $KE_{2n-2}^+$ spaces

In this section we will construct explicit examples of the geometries in  $2n + 1$  dimensions. For each Kähler-Einstein manifold with positive curvature in  $2n - 2$  dimensions,

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<sup>5</sup>Note that this equation first appeared, for  $n = 2$ , in a different context in [26].

the construction gives countably infinite classes of simply connected, compact geometries. The strategy is to first find local Kähler metrics satisfying (3.19) and then afterwards show that they lead to globally defined complete geometries in  $2n + 1$  dimensions. The approach is the analogue of the construction of Sasaki-Einstein manifolds in [8, 9] and generalises the analysis of [3] from  $n = 3, 4$  to all  $n \geq 3$ .

In order to find explicit examples of local Kähler metrics in  $2n$  dimensions satisfying (3.19), following [27], we consider the ansatz

$$ds_{2n}^2 = \frac{d\rho^2}{U} + U\rho^2(D\phi)^2 + \rho^2 ds^2(KE_{2n-2}^+) \quad (4.1)$$

with

$$D\phi = d\phi + C. \quad (4.2)$$

Here  $ds^2(KE_{2n-2}^+)$  is a  $2n - 2$ -dimensional Kähler-Einstein metric of positive curvature. It is normalised so that  $\mathcal{R}_{KE} = 2nJ_{KE}$  and the one-form  $C$  satisfies  $dC = 2J_{KE}$ . Note that  $nC$  is then the connection on the canonical bundle of the Kähler-Einstein space. Let  $\Omega_{KE}$  denote a local  $(n - 1, 0)$ -form, unique up to rescaling by a complex function.

To show that  $ds_{2n}^2$  is a Kähler metric observe that the Kähler form, defined by

$$J_T = \rho d\rho \wedge D\phi + \rho^2 J_{KE}, \quad (4.3)$$

is closed, and that the holomorphic  $(n, 0)$ -form

$$\Omega_T = e^{in\phi} \left( \frac{d\rho}{\sqrt{U}} + i\rho\sqrt{U}D\phi \right) \wedge \rho^{n-1}\Omega_{KE} \quad (4.4)$$

satisfies

$$d\Omega_T = ifD\phi \wedge \Omega_T, \quad (4.5)$$

with

$$f = n(1 - U) - \frac{\rho}{2} \frac{dU}{d\rho}. \quad (4.6)$$

This implies, in particular, that the complex structure defined by  $\Omega$  is integrable. In addition (4.5) allows us to obtain the Ricci tensor of  $ds_{2n}^2$ :

$$\mathcal{R} = dP, \quad P = fD\phi. \quad (4.7)$$

The Ricci-scalar is then obtained via  $R = \mathcal{R}_{ij}J^{ij}$ .

We would like to find the conditions on  $U$  such that  $ds_{2n}^2$  satisfies the equation (3.19). It is convenient to introduce the new coordinate  $x = 1/\rho^2$  so that

$$ds_{2n}^2 = \frac{1}{x} \left[ \frac{dx^2}{4x^2U} + U(D\phi)^2 + ds^2(KE_{2n-2}^+) \right] \quad (4.8)$$

and

$$f = n(1 - U) + x \frac{dU}{dx} \quad (4.9)$$

$$R = 4(n - 1)xf - 4x^2 \frac{df}{dx}. \quad (4.10)$$

We can now show that (3.19) can be integrated once to give

$$2(n - 1)f^2 + U \frac{dR}{dx} = (\text{constant}) \times x^{n-2}. \quad (4.11)$$

It is now straightforward to obtain polynomial solutions of (4.11). For simplicity we will restrict our considerations<sup>6</sup> to solutions of the form  $U = 1 - \alpha x^{n-2}(x - \beta)^2$ . Note that if we scale  $x \rightarrow kx$ , we obtain the same  $2n + 1$  dimensional metric (see below) providing that  $\alpha \rightarrow k^n \alpha$ ,  $\beta \rightarrow \beta/k$ . For reasons that will become clear soon, we are interested in  $U$  having two distinct roots. If  $n$  is even then we must have  $\alpha > 0$ . If  $n$  is odd, by rescaling  $x$  if necessary, we can also take  $\alpha > 0$ . We will also use this scaling to set  $\beta = 1$ . Thus we will focus on solutions with

$$U = 1 - \alpha x^{n-2}(x - 1)^2. \quad (4.12)$$

Observing that  $U$  has turning points at  $x = (n - 2)/n$  and at  $x = 1$ , we will choose  $\alpha \in (\alpha_0, \infty)$  where  $\alpha_0 = n^n/(4(n - 2)^{n-2})$  so that  $U$  has two positive roots  $x_1$  and  $x_2$ .

We now consider the local metrics in  $2n + 1$  dimensions that can be constructed from these local  $2n$ -dimensional Kähler metrics:

$$ds_{2n+1}^2 = c^2 \left[ (dz + P)^2 + \frac{R}{2} ds_{2n}^2 \right], \quad (4.13)$$

where  $R$  is the Ricci-scalar of the  $2n$ -dimensional metric given in (4.9):

$$R = 8\alpha x^{n-1}. \quad (4.14)$$

The scalar  $B$  and the two-form  $F$  can be obtained from (3.18). It will be very convenient to employ the coordinate transformation

$$\phi = \frac{1}{n}(\psi - z) \quad (4.15)$$

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<sup>6</sup>For  $n = 3$  and  $n = 4$ , see [3] for some discussion of other solutions.

so that the metric can be written

$$\frac{1}{c^2}ds_{2n+1}^2 = wDz^2 + \frac{RU}{2n^2wx}D\psi^2 + \frac{R}{8x^3U}dx^2 + \frac{R}{2x}ds^2(KE_{2n-2}^+) \quad (4.16)$$

where  $D\psi = d\psi + nB$ ,  $Dz = dz + gD\psi$  with

$$\begin{aligned} w &= \left(1 - \frac{f}{n}\right)^2 + \frac{RU}{2n^2x} \\ g &= \frac{1}{n^2w}(nf - f^2 - \frac{RU}{2x}). \end{aligned} \quad (4.17)$$

We demand that  $w > 0$ ,  $R > 0$  and  $U \geq 0$ . We will achieve this by demanding that  $x \in [x_1, x_2]$  where  $x_i$  are two positive roots of  $U(x)$ .

We now want to argue that this local metric, for countably infinite values of  $\alpha \in (\alpha_0, \infty)$  and for any positively curved Kähler-Einstein manifold, globally extends to give a complete, compact metric on a  $2n + 1$  dimensional manifold. We will argue this in two steps. We first study the  $2n$ -dimensional metric transverse to the  $z$  direction in (4.16) and then consider the  $U(1)$  fibration, with fibre parametrised by the coordinate  $z$ , which we will take to be periodic with a suitable chosen period.

We start by analysing the  $2n$ -dimensional metric transverse to the  $z$  direction in (4.16). As we have already mentioned we take  $x \in [x_1, x_2]$  and we take  $\psi$  to be a periodic co-ordinate with period  $2\pi$ . Then the above  $2n$ -dimensional metric extends to a smooth complete metric on the total space of an  $S^2$  bundle over the original Kähler-Einstein space,  $KE_{2n-2}^+$ , and this is true for any value of  $\alpha \in (\alpha_0, \infty)$ . To see this, we observe that the two sphere is parametrised by  $\psi, x$ . The key issue is to ensure that the metric has no conical singularities at the poles of the two-sphere which are located at  $x = x_1, x_2$ . A small calculation show that since at any root  $x_i$  of  $U$  we have

$$\frac{(U'x)^2}{w}|_{x=x_i} = n^2, \quad (4.18)$$

there will be no conical singularities if we take  $\psi$  to have period  $2\pi$ . Let us call this  $2n$ -dimensional manifold  $Y_{2n}$ .

We now turn to the  $2n + 1$ -dimensional metric. The idea is to choose  $z$  to have period  $2\pi l$ , for suitable  $l$ , so that the metric is that of the total space of a  $U(1)$  fibration over the globally defined  $2n$  dimensional base manifold  $Y_{2n}$ , with connection one-form given by  $l^{-1}gD\psi$ . The key point here is to ensure that the periods of the  $(1/2\pi l)d(gD\psi)$  over a basis for the free part of the second homology group of  $Y_{2n}$  are integers. This is an almost identical set up to that discussed in [9] and we refer to that paper for more details. The result, in order to obtain a simply connected



manifold, is that we need to choose

$$\frac{g(x_2)}{g(x_1)} = \frac{p}{q}, \quad (4.19)$$

where  $p, q$  are relatively prime integers. We also choose

$$l = \frac{hg(x_1)}{q}, \quad (4.20)$$

where  $h$  is the highest common factor of  $p - q$  and  $qc_i$  where  $c_i$  are the Chern numbers of the Kähler-Einstein manifold  $KE_{2n-2}$ . Finally, we will show below that as  $\alpha$  ranges from  $\alpha_0$  to  $\infty$ ,  $\frac{g(x_2)}{g(x_1)}$  monotonically increases from 0 to 1. Hence there will be countably infinite values of  $\alpha$  that satisfy (4.19) and hence countably infinite metrics on complete  $2n + 1$ -dimensional manifolds.

To examine the behaviour of  $g(x_2)/g(x_1)$  as a function of  $\alpha$  we first note that using  $U(x_i) = 0$  we obtain the simple expression

$$g(x_i) = -\frac{2}{n(x_i - 1) + 2}. \quad (4.21)$$

To proceed, we recall that the turning points of  $U(x)$  are at  $x = (n - 2)/n$  and at  $x = 1$  and hence  $(n - 2)/n < x_1 < 1 < x_2$ . We thus conclude that  $g(x_i)$  is negative for both  $x_1$  and  $x_2$ . We now observe that

$$\frac{d}{d\alpha}g(x_i) = \frac{2n}{(n(x_i - 1) + 2)^2} \frac{dx_i}{d\alpha}. \quad (4.22)$$

Next, since  $U(x_i) = 0$ , we have

$$\alpha = \frac{1}{x_i^{n-2}(x_i - 1)^2} \quad (4.23)$$

and we can compute

$$\frac{dx_i}{d\alpha} = -\frac{x_i^{n-1}(x_i - 1)^3}{n(x_i - 1) + 2} \quad (4.24)$$

This is negative for  $x_2$ , and positive for  $x_1$  and hence we deduce the required monotonicity property of  $g(x_2)/g(x_1)$ .

## 5 An ansatz for the geometry in $2n + 2$ dimensions

Geometries in  $2n + 1$ -dimensions that can be constructed from a  $2n$ -dimensional Kähler manifold consisting of a product of Kähler-Einstein manifolds satisfying (3.19) were studied in [3]. As we have discussed these give rise to conical geometries in

$2n + 2$ -dimensions with an  $SU(n + 1)$  structure satisfying (1.3), (1.4). In this section we will consider an ansatz for the geometry in  $2n + 2$ -dimensions that generalises these conical geometries. We will derive the ordinary differential equations that need to be solved and while we have not been able to find the most general solution, we do recover some known solutions.

Our metric ansatz is given by

$$ds_{2n+2}^2 = \alpha^2 dr^2 + \beta^2 (dz + P)^2 + \sum_{i=1}^n \gamma_i^2 ds_i^2(KE_2), \quad (5.1)$$

where  $ds_i^2(KE_2)$  denotes the metric of the  $i$ th two-dimensional Kähler-Einstein space. We will let  $J_i$ ,  $\Omega_i$  and  $\mathcal{R}_i$  denote the Kähler-form, the  $(1, 0)$  form and the Ricci-form of the corresponding Kähler-Einstein space, respectively. We also have  $P = \sum P_i$  where

$$dP_i = \mathcal{R}_i = l_i J_i \quad (5.2)$$

and  $l_i$  are constant. Note that if we set some of  $l_i$ 's to the same value, and also set the corresponding  $\gamma_i$ 's to be equal, one can also replace the relevant product of the two-dimensional Kähler-Einstein spaces with higher-dimensional Kähler-Einstein spaces.

The  $SU(n + 1)$  structure  $J, \Omega$  is given by

$$J = -\alpha\beta dr \wedge (dz + P) + \sum_i \gamma_i^2 J_i \quad (5.3)$$

$$\Omega = [\alpha dr - i\beta(dz + P)] \wedge \prod_i \gamma_i \Omega_i. \quad (5.4)$$

We will demand that the  $SU(n + 1)$  structure satisfies (1.3), (1.4) which we write again here:

$$d(e^{n\phi}\Omega) = 0 \quad (5.5)$$

$$d(e^{2(n-1)\phi}J^n) = 0 \quad (5.6)$$

$$d[e^{2(n-3)\phi} *_{2n+2} d(e^{2\phi}J)] = 0. \quad (5.7)$$

If we set  $e^\phi = \lambda$  and assume that  $\alpha, \beta, \gamma_i$  and  $\lambda$  are all functions of  $r$ , we obtain the following set of coupled nonlinear ordinary differential equations:

$$\alpha\gamma\lambda^n + (\beta\gamma\lambda^n)' = 0 \quad (5.8)$$

$$\alpha\beta\gamma^2\lambda^{2(n-1)} \sum \frac{l_i}{\gamma_i^2} + (\gamma^2\lambda^{2(n-1)})' = 0 \quad (5.9)$$

$$\frac{\beta\gamma^2\lambda^{2(n-3)}}{\alpha\gamma_i^4} [l_i\alpha\beta\lambda^2 + (\gamma_i^2\lambda^2)'] = k_i \quad (5.10)$$

$$\sum k_i l_i = 0, \quad (5.11)$$

where  $k_i$  are constants and  $\gamma = \prod \gamma_i$ .

To recover the simple metric cone geometries whose  $2n$ -dimensional base geometries were discussed in [3], we start with

$$\alpha = 1, \quad \beta = \gamma_i = cr, \quad \lambda = r^k. \quad (5.12)$$

The above equations then give

$$c = \frac{n-2}{2}, \quad k = -\frac{n-1}{n-2}, \quad k_i = c^{2(n-1)}(l_i - 1) \quad (5.13)$$

and

$$\sum_i l_i = \sum_i l_i^2 = 1. \quad (5.14)$$

The base of this metric cone is the  $2n + 1$ -dimensional geometry built from a  $2n$ -dimensional Kähler base consisting of a product of Kähler-Einstein spaces. The last equation, which up to a scaling was noted in [3], is the algebraic form of the master equation (3.19) for the Kähler base consisting of a product of Kähler-Einstein spaces.

Let us now construct some other solutions for  $n = 3$  and  $n = 4$  which give rise to supersymmetric solutions of type IIB supergravity and  $D = 11$  supergravity via (2.12) and (2.13), respectively. We first consider the  $n = 3$  case. The solution describing the  $AdS_3$  limit of D3-branes wrapped on  $H_2/\Gamma$ , a Riemann surface with genus  $g > 1$ , in a Calabi-Yau four-fold [11, 12, 28, 29], which was discussed from the present point of view in [3], can be recovered by setting

$$\begin{aligned} l_1 &= -1/3, & l_2 = l_3 &= 2/3, \\ k_1 &= -1/12, & k_2 = k_3 &= -1/48 \end{aligned} \quad (5.15)$$

and

$$\alpha = 1, \quad \beta = \gamma_i = r/2, \quad \lambda = 1/r^2. \quad (5.16)$$

Since  $l_2 = l_3$  and  $\gamma_2 = \gamma_3$  we can replace the corresponding  $KE_2 \times KE_2$  with a  $KE_4$  space. For simplicity we will just discuss the case when  $KE_4 = CP^2$ .

We would like to know whether there exists a more general solution to the above coupled nonlinear differential equation, with the same parameters  $l_i, k_i$ . One can

indeed find that

$$\alpha^2 = \left(1 - \frac{m}{r^{4/3}}\right)^{-1} \quad (5.17)$$

$$\beta^2 = \frac{r^2}{4} \left(1 - \frac{m}{r^{4/3}}\right) \quad (5.18)$$

$$\gamma_1^2 = \frac{r^2}{4} \quad (5.19)$$

$$\gamma_2^2 = \gamma_3^2 = \frac{r^2}{4} \left(1 - \frac{m}{r^{4/3}}\right) \quad (5.20)$$

$$\lambda^{-1} = r^2 \left(1 - \frac{m}{r^{4/3}}\right), \quad (5.21)$$

satisfies the equations, where  $m$  is a constant. Using (2.12) we find that the ten dimensional type IIB metric takes the form

$$\begin{aligned} ds^2 = & \frac{1}{r^2(1 - \frac{m}{r^{4/3}})} ds^2(\mathbb{R}^{1,1}) + \frac{1}{4(1 - \frac{m}{r^{4/3}})} ds^2(H_2/\Gamma) + \frac{dr^2}{r^2(1 - \frac{m}{r^{4/3}})^2} \\ & + \frac{1}{4} [(dz + P)^2 + ds^2(CP^2)]. \end{aligned} \quad (5.22)$$

If we take  $m > 0$ ,  $0 \leq r \leq m^{3/4}$  we essentially have the solution of [11, 12]. Note that, as discussed in section 6.1 of [29], we can choose the range of  $z$  to be  $6\pi$  if  $g-1$  is divisible by three and  $2\pi$  otherwise. In the former case, the solution interpolates between a locally  $AdS_5 \times S^5$  region and the  $AdS_3 \times H_2/\Gamma \times S^5$  solution given above in (5.16). In the latter case the  $S^5$  is replaced with  $S^5/Z_3$ .

The above solution can be interpreted as describing the near horizon limit of a D3-brane wrapping a holomorphic  $H_2/\Gamma$  inside a Calabi-Yau four-fold ( $CY_4$ ) [12]. It would be interesting if we could find a solution that interpolated from an asymptotic  $CY_4$  region to this solution or perhaps just to the  $AdS_3 \times H_2/\Gamma$  solution (5.16). We have not been able to construct such a solution, but we observe that our ansatz does include the following Calabi-Yau four-fold metric

$$ds^2 = \frac{1}{U} dr^2 + \frac{9}{16} U r^2 (dz + P_1 + P_2)^2 + \frac{l_1 3}{8} (r^2 + c) ds_1^2(KE_2) + \frac{9}{4} r^2 ds_2^2(CP^2), \quad (5.23)$$

where

$$U = \frac{3r^2 + 4c}{9(r^2 + c)}, \quad (5.24)$$

we have normalised  $ds^2(KE_2)$  so that  $l_1 = \pm 1$  and we have taken  $KE_4$  to be  $CP^2$  with  $l_2 = 6$ . When  $l_1 = 1$ ,  $KE_2$  is a unit radius  $S^2$ . Choosing  $c > 0$  and  $0 \leq r < \infty$  we have the metric of [10] (eq. (5.34)). In particular the range of  $z$  is  $2\pi$  and as  $r \rightarrow \infty$  the metric is asymptotically a cone over a regular Sasaki-Einstein space,

while at  $r = 0$  we have an  $S^2$  bolt: note that since the period of  $z$  is  $2\pi$  at  $r = 0$  the  $CP^2$  and the  $z$  fibre combine to give  $S^5/Z_3$  and so there is a conical singularity.

On the other hand when  $l_1 = -1$ , the case of more relevance here, we choose  $KE_2 = H/\Gamma$ , again a Riemann surface with genus  $g > 1$ . We also take  $c < 0$  and  $0 \leq r^2 < c$ . We can choose the range of  $z$  to be  $6\pi$  if  $g - 1$  is divisible by three and  $2\pi$  otherwise. In the former case, there is no conical singularity at the  $H^2/\Gamma$  bolt at  $r = 0$ , while in the latter case there is. Note that this metric is singular as  $r^2 \rightarrow -c$ . This metric provides a natural local model of a holomorphic  $H_2/\Gamma$  in a  $CY_4$  for which  $D3$ -branes can wrap.

Now let us consider the M-theory case with  $n = 4$  which is very similar. Setting

$$\begin{aligned} l_1 &= -1/2, & l_2 = l_3 = l_4 &= 1/2, \\ k_1 &= -3/2, & k_2 = k_3 = k_4 &= -1/2, \end{aligned} \quad (5.25)$$

we obtain the solution

$$\alpha = 1, \quad \beta = \gamma_i = r, \quad \lambda = r^{-3/2}, \quad (5.26)$$

which corresponds to the  $AdS_2$  limit of M2-branes wrapping a holomorphic  $H^2/\Gamma$ , again a Riemann surface of genus  $g > 1$ , in a Calabi-Yau five-fold [13, 14]. We can also find a more general solution, with

$$\alpha^2 = \frac{1}{1 - \frac{m}{r}} \quad (5.27)$$

$$\beta^2 = \left(1 - \frac{m}{r}\right) r^2 \quad (5.28)$$

$$\gamma_1^2 = r^2 \quad (5.29)$$

$$\gamma_2^2 = \gamma_3^2 = \gamma_4^2 = \left(1 - \frac{m}{r}\right) r^2 \quad (5.30)$$

$$\lambda^{-4/3} = \left(1 - \frac{m}{r}\right) r^2. \quad (5.31)$$

The corresponding  $D = 11$  metric can be easily constructed from (2.13) and is given by

$$ds^2 = -\frac{1}{r^2(1 - m/r)} dt^2 + \frac{1}{1 - m/r} ds^2(H_2/\Gamma) + \frac{dr^2}{r^2(1 - m/r)^2} \quad (5.32)$$

$$+ (dz + P)^2 + ds^2(CP^3). \quad (5.33)$$

Where, for simplicity, we have restricted attention to the case of  $KE_6 = CP^3$ . If we take  $m > 0$ ,  $0 \leq r \leq m$  we essentially have the solution of [13, 14]. Note that we can choose the range of  $z$  to be  $8\pi$  if  $g$  is odd and  $4\pi$  if  $g$  is even. In the former case, the

solution interpolates between a locally  $AdS_4 \times S^7$  region and the  $AdS_2 \times H_2/\Gamma \times S^7$  solution given above (5.26). In the latter case the  $S^7$  is replaced with  $S^7/Z_2$ .

Again it would be interesting if we could find a solution that interpolated from an asymptotic  $CY_5$  region to this solution or perhaps just to the  $AdS_2 \times H_2/\Gamma$  solution (5.26). We have not been able to construct such a solution, but we observe that our ansatz does include the following Calabi-Yau five-fold metric

$$ds^2 = \frac{80}{U} dr^2 + \frac{U}{5} r^2 (dz + P_1 + P_2)^2 + l_1 2(r^2 + c) ds_1^2(H_2/\Gamma) + 16r^2 ds_2^2(CP^3) \quad (5.34)$$

where

$$U = \frac{4r^2 + 5c}{(r^2 + c)}, \quad (5.35)$$

we have normalised  $ds^2(KE_2)$  so that  $l_1 = \pm 1$  and we have taken  $KE_6$  to be  $CP^3$  with  $l_2 = 8$ . When  $l_1 = 1$ ,  $KE_2$  is a unit radius  $S^2$ . Choosing  $c > 0$  and  $0 \leq r < \infty$  we have the metric in the general class of [10]. The range of  $z$  is  $4\pi$  and as  $r \rightarrow \infty$  the metric is asymptotically a cone over a regular Sasaki-Einstein space, while at  $r = 0$  we have an  $S^2$  bolt: note that since the period of  $z$  is  $4\pi$  at  $r = 0$  the  $CP^3$  and the  $z$  fibre combine to give  $S^7/Z_2$  and so there is a conical singularity. On the other hand when  $l_1 = -1$ , the case of more relevance here, we take  $KE_2 = H/\Gamma$ ,  $c < 0$  and  $0 \leq r^2 < c$ . We can now choose the range of  $z$  to be  $8\pi$  if  $g$  is odd and  $4\pi$  if  $g$  is even. In the former case, there is no conical singularity at the  $H^2/\Gamma$  bolt at  $r = 0$ , while in the latter case there is. Note that this metric is singular as  $r^2 \rightarrow -c$ , but nevertheless provides a good local model of a holomorphic  $H_2/\Gamma$  embedded in a  $CY_5$ , for which membranes can wrap.

## 6 LLM inspired ansatz

In this section we consider an ansatz that is motivated by the results of Lin, Lunin and Maldacena (LLM) [15]. It was shown in [1] and [2] how one can recast the results of LLM in terms of a local Kähler geometry in  $2n$  dimensions satisfying (1.1), for  $n = 3$  and  $n = 4$ . Here we extend this by constructing an ansatz for general  $n$ .

We start with the following ansatz for the local  $2n$ -dimensional Kähler metric

$$ds^2 = \frac{dy^2}{U} + y^2 U (D\psi)^2 + \frac{f}{U} (dx_1^2 + dx_2^2) + y^2 ds^2(KE_{2n-4}), \quad (6.1)$$

where  $ds^2(KE_{2n-4})$  is a Kähler-Einstein metric (possibly local),  $D\psi = d\psi + \sigma + V$ ,  $\sigma$  is a one-form on the Kähler-Einstein space,  $V$  is a one-form on the two-dimensional space spanned by  $x_i$ ,  $i = 1, 2$  and  $U, f, V$  all depend on three coordinates  $y, x_i$ .

We take the Kähler form to be given by

$$J_T = ydy \wedge D\psi + \frac{f}{U} dx_1 \wedge dx_2 + y^2 J_{KE}, \quad (6.2)$$

where  $J_{KE}$  is the Kähler form on the Kähler-Einstein space. Then, demanding that  $dJ_T = 0$  we obtain the following conditions

$$d\sigma = 2J_{KE} \quad (6.3)$$

$$d_2 V = \frac{1}{y} \partial_y \left( \frac{f}{U} \right) dx_1 \wedge dx_2, \quad (6.4)$$

where  $d_2 \equiv dx^i \wedge \partial_i$ . In order to see if the complex structure is indeed integrable we need to compute the derivative of the  $(n, 0)$ -form  $\Omega_T$  given by

$$\Omega_T = e^{ik\psi} \left( \frac{dy}{\sqrt{U}} + iy\sqrt{U} D\psi \right) \wedge \sqrt{\frac{f}{U}} (dx_1 + idx_2) \wedge y^{n-2} \Omega_{KE}, \quad (6.5)$$

where  $\Omega_{KE}$  is the  $(n-2, 0)$ -form on  $KE_{2n-4}$  which satisfies

$$d\Omega_{KE} = ik\sigma \wedge \Omega_{KE}. \quad (6.6)$$

The constant  $k$  determines the normalisation of the  $KE_{2n-4}$  space. In particular, we have  $\mathcal{R}_{KE} = 2kJ_{KE}$ , where  $\mathcal{R}_{KE}$  is the Ricci-form of the KE space. We now find that

$$d\Omega = iP \wedge \Omega, \quad (6.7)$$

with the Ricci potential given by

$$P = \frac{1}{2} *_2 d_2 (\ln f) - \frac{y^{2-n} U}{\sqrt{f}} \partial_y \left( y^{n-1} \sqrt{f} \right) D\psi + k(d\psi + \sigma), \quad (6.8)$$

provided that we impose

$$\partial_y V = \frac{1}{y} *_2 d_2 \left( \frac{1}{U} \right). \quad (6.9)$$

The compatibility of (6.4) and (6.9) leads to the following equation:

$$\Delta \left( \frac{1}{U} \right) + y \partial_y \left[ \frac{1}{y} \partial_y \left( \frac{f}{U} \right) \right] = 0, \quad (6.10)$$

where  $\Delta = \partial_1^2 + \partial_2^2$ .

Having obtained the Ricci potential, the next step would be to compute the Ricci tensor and see how the master equation (1.1) can be satisfied. To simplify things, we first introduce a function  $D$  defined via

$$\frac{1}{U} = \frac{y}{2} \partial_y D. \quad (6.11)$$

We can now readily integrate (6.9) to get

$$V_i = \frac{1}{2} \epsilon_{ij} \partial_j D, \quad i, j = 1, 2. \quad (6.12)$$

In terms of  $D$ , (6.4) is now expressed as

$$\Delta D + \frac{1}{y} \partial_y (f y \partial_y D) = 0. \quad (6.13)$$

Furthermore, based on hints from [15] in the  $n = 3, 4$  case, we now make the assumption that there exists a relation

$$f = y^{2p} e^{qD}, \quad (6.14)$$

which makes the master equation (1.1) identically satisfied. Here  $p, q$  are constants that are to be fixed in terms of other parameters  $n, k$  that we have already introduced.

Noting that now

$$\frac{1}{2} *_2 d_2 \ln f = qV, \quad (6.15)$$

we can rewrite the Ricci potential  $P$  as

$$P = [k - q - (n + p - 1)U] D\psi + (q - k)V. \quad (6.16)$$

It is now straightforward to obtain the Ricci-form  $dP$  and from that the Ricci scalar which takes the simple form

$$R = \frac{4}{y^2} [(k - q)(n - 2) - q(n + p - 1) - (n + p - 1)(n + p - 2)U]. \quad (6.17)$$

After some computation one can now check that (1.1) is satisfied, if we demand that

$$p = 3 - n \quad (6.18)$$

and

$$(n - 2)(q - k)[q(n - 1) - k(n - 3)] = 0. \quad (6.19)$$

Let us first discuss the special case when  $n = 3$ . In this case we can solve the equations by taking  $p = q = 0$ . Then the only equation that needs to be solved is the linear equation

$$\Delta D + \frac{1}{y} \partial_y (y \partial_y D) = 0 \quad (6.20)$$

and we have recovered the equation of LLM for the type IIB case<sup>7</sup>

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<sup>7</sup>In order to recover the LLM result one, when one adds in the extra coordinate  $z$  to obtain a seven manifold. One should also shift the  $\psi$  coordinate  $\psi \rightarrow \psi + \alpha z$  for some constant  $\alpha$ .



For generic values of  $n \geq 3$ , the second equation is satisfied if  $q = k$ . Since  $k$  is related to the scalar curvature of the Kähler-Einstein base we can rescale it to  $0, \pm 1$  without losing generality. Here let us assume that  $k \neq 0$ . Then the entire solution is governed by the equation (when  $k = -1$  we redefine  $D \rightarrow -D$ )

$$\Delta D + \frac{1}{y} \partial_y (y^{7-2n} \partial_y e^D) = 0. \quad (6.21)$$

After the coordinate change  $x = (2n - 6)^{2(3-n)/(n-2)} y^{2n-6}$ , this equation becomes

$$\Delta D + x^{\frac{n-4}{n-3}} \partial_x^2 e^D = 0. \quad (6.22)$$

When  $n = 4$  this equation is the continuous Toda equation just as LLM discovered in the context of  $D = 11$  supergravity [15]. We do not know whether or not (6.21) is also an integrable system for  $n > 4$ . But it is at least clear that (6.21) still enjoys the 2d conformal symmetry, i.e. the equation is invariant under transformation

$$x_1 + ix_2 \rightarrow g(x_1 + ix_2), \quad D \rightarrow D - \log |\partial g|^2 \quad (6.23)$$

for an analytic function  $g$ .

## 7 Conclusions

In this paper we have introduced a new class of geometries in  $2n + 2$  dimensions that are specified by a metric, a scalar and a three-form. The geometries admit a specific kind of Killing spinor or, equivalently, a specific kind of  $SU(n + 1)$  structure. For  $n = 3$  and  $n = 4$  these give rise to supersymmetric solutions of type IIB and  $D = 11$  supergravity, with  $\mathbb{R}^{1,1}$  and  $\mathbb{R}$  factors, respectively.

We also showed that if these geometries in  $2n + 2$  dimensions are a certain kind of metric cone, then we obtain a new class of metric contact geometries in  $2n + 1$  dimensions, on the base of the cone, that are specified by a metric, a scalar and a two-form. For  $n = 3$  and  $n = 4$  these give rise to supersymmetric solutions of type IIB and  $D = 11$  supergravity, with  $AdS_3$  and  $AdS_2$  factors, that were discussed in [1] and [2], respectively. We have noted the strong similarities with Ricci-flat Kähler cones and Sasaki-Einstein manifolds.

We also constructed some specific examples of these geometries in sections 4-6. The constructions in section 4 can be straightforwardly extended by generalising the construction of section 5 of [3]. It should also be possible to extend this construction further using the results of [30, 25]. In sections 5 and 6 of this paper our constructions boiled down to solving some differential equations and we think it would be worthwhile to try and find additional solutions.

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